STA 360/602L: Module 3.5

The normal model: joint inference for mean and variance

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STA 602L

Joint inference for mean and variance

- We have derived the posterior for the μ, conditional on $\sigma/\ \tau$ being known. What happens when $\sigma/\ \tau$ is unknown? We need a joint prior $\pi(\mu, \sigma^2)$ for μ and σ^2.

- Write the joint prior distribution for the mean and variance as the product of a conditional and a marginal distribution. That is,

$$\pi(\mu, \sigma^2) \ = \ \pi(\mu|\sigma^2)\pi(\sigma^2).$$

- From the previous module, we have seen that we can set the conditional prior $\pi(\mu|\sigma^2)$ to be a normal distribution.

- For $\pi(\sigma^2)$, we need a distribution with support on $(0, \infty)$. One such family is the gamma family, but this is NOT conjugate for the variance of a normal distribution.

- The gamma distribution is, however, conjugate for the precision τ, and in that case, we say that σ^2 has an inverse-gamma distribution.

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Joint inference for mean and variance

- Recall that conjugacy means that for a prior $\pi(\theta)$ in a class of distributions $\mathcal{P}$, $\pi(\theta|Y)$ is also in class $\mathcal{P}$.

- However, when we have multiple parameters, the dependence structure in the prior must also be preserved in the posterior, for conjugacy to hold.

- So, if

$$\pi(\mu, \sigma^2) \;=\; \pi(\mu|\sigma^2)\pi(\sigma^2).$$

with $\pi(\mu|\sigma^2)$ a normal distribution, and $\pi(\sigma^2)$ an inverse-gamma distribution, we will have conjugacy if $\pi(\mu, \sigma^2|Y)$ can also be written as

$$\pi(\mu, \sigma^2|Y) \;=\; \pi(\mu|\sigma^2, Y)\pi(\sigma^2|Y),$$

where $\pi(\mu|\sigma^2, Y)$ is also a normal distribution, and $\pi(\sigma^2|Y)$ is an inverse-gamma distribution, just like the prior.

Inverse-gamma distribution

- As before, we will continue to work mostly in terms of the precision τ.
- That is, we will deal with the already familiar gamma distribution, instead of the inverse-gamma distribution.
- However, as a quick review, if $\theta \sim \mathcal{IG}(a, b)$, then the pdf is

$$p(\theta) = \frac{b^a}{\Gamma(a)} \theta^{-(a+1)} e^{-\frac{b}{\theta}} \quad \text{for} \quad a, b > 0,$$

where

  - $\mathbb{E}[\theta] = \frac{b}{a-1}$;
  - $\mathbb{V}[\theta] = \frac{b^2}{(a-1)^2(a-2)} \;\; \text{for} \;\; a \geq 2.$;
  - $\text{Mode}[\theta] = \frac{b}{a+1}$.

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Conjugate prior

- Once again, suppose $Y = (y_1, y_2, \ldots, y_n)$, where each

$$y_i \sim \mathcal{N}(\mu, \tau^{-1}).$$

- A conjugate joint prior is given by

$$\tau = \frac{1}{\sigma^2} \sim \text{Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2}\right)$$

$$\mu | \tau \sim \mathcal{N}\left(\mu_0, \frac{1}{\kappa_0 \tau}\right).$$

- This is often called a normal-gamma prior distribution.
- σ_0^2 is the prior guess for σ^2, while ν_0 is often referred to as the "prior degrees of freedom", our degree of confidence in σ_0^2.
- We do not have conjugacy if we replace $\frac{1}{\kappa_0 \tau}$ in the normal prior with an arbitrary prior variance independent of τ / σ^2. To do inference in that scenario, we need Gibbs sampling (to come soon!).

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Conjugate prior

- So, we have

$$\pi(\mu|\tau) = \mathcal{N}\left(\mu_0, \frac{1}{\kappa_0 \tau}\right) \propto \tau^{\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}\kappa_0 \tau (\mu - \mu_0)^2\right\}.$$

- and

$$\pi(\tau) = \text{Ga}\left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2}\right) \propto \tau^{\frac{\nu_0}{2}-1} \exp\left\{-\frac{\tau \nu_0 \sigma_0^2}{2}\right\}.$$

- Thus, the kernel of the normal-gamma prior distribution is

$$\begin{aligned} \Rightarrow \pi(\mu, \tau) &= \pi(\mu|\tau) \cdot \pi(\tau) = \mathcal{N}\left(\mu_0, \frac{1}{\kappa_0 \tau}\right) \cdot \text{Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0 \sigma_0^2}{2}\right) \\ &\propto \underbrace{\tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\kappa_0 \tau (\mu - \mu_0)^2\right\}}_{\propto \pi(\mu|\tau)} \cdot \underbrace{\tau^{\frac{\nu_0}{2}-1} \exp\left\{-\frac{\tau \nu_0 \sigma_0^2}{2}\right\}}_{\propto \pi(\tau)}. \end{aligned}$$

- Take note of this form. When we derive the posterior kernel, we will try to match it to this to recognize the parameters.

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Posterior for the mean given variance, under normal-gamma prior

- Based on the normal-gamma prior, we need $\pi(\mu|Y,\tau)$ and $\pi(\tau|Y)$.
- For $\pi(\mu|Y,\tau)$, we already know from the previous module that it will be a normal distribution.
- However, some algebra is required to get $\pi(\tau|Y)$.
- Infact, we need to write the full joint posterior and go from there, because we will need to keep some of the terms we discarded in the derivation in the last module.
- First, recall that the likelihood is

$$P(Y|\mu,\tau) \propto \tau^{\frac{n}{2}} \exp\left\{-\frac{1}{2}\tau s^2(n-1)\right\} \exp\left\{-\frac{1}{2}\tau n(\mu-\bar{y})^2\right\}.$$

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Posterior derivation

$$\text{Then,} \quad \pi(\mu,\tau|Y) \propto \pi(\mu|\tau)\times\pi(\tau)\times P(Y|\mu,\tau)$$

$$\propto \underbrace{\tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\kappa_0\tau(\mu-\mu_0)^2\right\}}_{\propto\,\pi(\mu|\sigma^2)} \times \underbrace{\tau^{\frac{\nu_0}{2}-1}\exp\left\{-\frac{\tau\nu_0\sigma_0^2}{2}\right\}}_{\propto\,\pi(\tau)}$$

$$\times \underbrace{\tau^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\tau s^2(n-1)\right\}\exp\left\{-\frac{1}{2}\tau n(\mu-\bar{y})^2\right\}}_{\propto\,P(Y|\mu,\tau)}$$

$$= \underbrace{\exp\left\{-\frac{1}{2}\kappa_0\tau(\mu-\mu_0)^2\right\}\exp\left\{-\frac{1}{2}\tau n(\mu-\bar{y})^2\right\}}_{\text{Terms involving } \mu}$$

$$\times \underbrace{\tau^{\frac{1}{2}}\tau^{\frac{\nu_0}{2}-1}\exp\left\{-\frac{\tau\nu_0\sigma_0^2}{2}\right\}\tau^{\frac{n}{2}}\exp\left\{-\frac{1}{2}\tau s^2(n-1)\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

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Posterior derivation

$$\pi(\mu,\tau|Y) \propto \underbrace{\exp\left\{-\frac{1}{2}\kappa_0\tau(\mu^2-2\mu\mu_0+\mu_0^2)\right\}\exp\left\{-\frac{1}{2}\tau n(\mu^2-2\mu\bar{y}+\bar{y}^2)\right\}}_{\text{Terms involving } \mu}$$

$$\times \underbrace{\tau^{\frac{1}{2}}\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

$$= \underbrace{\exp\left\{-\frac{1}{2}\left[\kappa_0\tau(\mu^2-2\mu\mu_0)+\tau n(\mu^2-2\mu\bar{y})\right]\right\}}_{\text{Terms involving } \mu}$$

$$\times \underbrace{\tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2+\tau n\bar{y}^2\right]\right\}\cdot\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

$$= \underbrace{\exp\left\{-\frac{1}{2}\left[\mu^2(n\tau+\kappa_0\tau)-2\mu(n\tau\bar{y}+\kappa_0\tau\mu_0)\right]\right\}}_{\text{Terms involving } \mu}$$

$$\times \underbrace{\tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2+\tau n\bar{y}^2\right]\right\}\cdot\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

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Posterior derivation

- To match the terms for the terms involving μ to the normal kernel in the prior, we need to complete the square so that we have something that looks like the $(\mu - \mu_0)^2$ term in our prior.

- Recall how to complete the square. Specifically, we can write

$$a\mu^2 + b\mu$$

as

$$a(\mu + d)^2 + e,$$

where

  - $d = \frac{b}{2a}$, and
  - $e = -\frac{b^2}{4a}$.

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Posterior derivation

- First, write out the posterior again:

$$\pi(\mu,\tau|Y) = \underbrace{\exp\left\{-\frac{1}{2}\left[(n\tau+\kappa_0\tau)\mu^2 - 2\mu(n\tau\bar{y}+\kappa_0\tau\mu_0)\right]\right\}}_{\text{Terms involving } \mu}$$
$$\times \underbrace{\tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2+\tau n\bar{y}^2\right]\right\} \cdot \tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

- Set $a^\star = (n\tau+\kappa_0\tau)$ and $b^\star = (n\tau\bar{y}+\kappa_0\tau\mu_0)$, then complete the square for the first part.

$$\Rightarrow \pi(\mu,\tau|Y) \propto \underbrace{\exp\left\{-\frac{1}{2}\left[a^\star\mu^2 - 2b^\star\mu\right]\right\}}_{\text{Terms involving } \mu}$$
$$\times \underbrace{\tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2+\tau n\bar{y}^2\right]\right\} \cdot \tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}}_{\text{Terms involving } \tau \text{ but NOT } \mu}$$

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Posterior derivation

$$\Rightarrow \pi(\mu,\tau|Y) \propto \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}a^\star\left[\mu - \frac{b^\star}{a^\star}\right]^2 + \frac{(b^\star)^2}{2a^\star}\right\} \cdot \exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2 + \tau n\bar{y}^2\right]\right\}$$
$$\times \tau^{\frac{\nu_0+n}{2}-1} \exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2 + s^2(n-1)\right]}{2}\right\}$$

$$= \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}a^\star\left[\mu - \frac{b^\star}{a^\star}\right]^2\right\} \underbrace{\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2 + \tau n\bar{y}^2 - \frac{(b^\star)^2}{a^\star}\right]\right\}}_{\text{Next, substitute the values for } a^\star \text{ and } b^\star \text{ back}}$$
$$\times \tau^{\frac{\nu_0+n}{2}-1} \exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2 + s^2(n-1)\right]}{2}\right\}$$

$$= \tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}a^\star\left[\mu - \frac{b^\star}{a^\star}\right]^2\right\} \underbrace{\exp\left\{-\frac{1}{2}\left[\kappa_0\tau\mu_0^2 + \tau n\bar{y}^2 - \frac{(n\tau\bar{y} + \kappa_0\tau\mu_0)^2}{(n\tau + \kappa_0\tau)}\right]\right\}}_{\text{Next, expand terms and recombine}}$$
$$\times \tau^{\frac{\nu_0+n}{2}-1} \exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2 + s^2(n-1)\right]}{2}\right\}$$

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Posterior derivation

$$
\begin{aligned}
\Rightarrow \pi(\mu,\tau|Y) \propto\; & \tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}a^\star\left[\mu-\frac{b^\star}{a^\star}\right]^2\right\}\exp\left\{-\frac{1}{2}\left[\frac{n\kappa_0\tau^2(\mu_0^2-2\mu_0\bar{y}+\bar{y}^2)}{\tau(\kappa_0+n)}\right]\right\}\\
& \times\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}\\
=\; & \tau^{\frac{1}{2}}\exp\left\{-\frac{1}{2}a^\star\left[\mu-\frac{b^\star}{a^\star}\right]^2\right\}\exp\left\{-\frac{\tau}{2}\left[\frac{n\kappa_0(\bar{y}-\mu_0)^2}{(\kappa_0+n)}\right]\right\}\\
& \times\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}\\
=\; & \tau^{\frac{1}{2}}\exp\underbrace{\left\{-\frac{1}{2}a^\star\left[\mu-\frac{b^\star}{a^\star}\right]^2\right\}}_{\text{Substitute the values for } a^\star \text{ and } b^\star \text{ back}}\\
& \times\tau^{\frac{\nu_0+n}{2}-1}\exp\left\{-\frac{\tau\left[\nu_0\sigma_0^2+s^2(n-1)\right]}{2}\right\}\exp\left\{-\frac{\tau}{2}\left[\frac{n\kappa_0(\bar{y}-\mu_0)^2}{(\kappa_0+n)}\right]\right\}
\end{aligned}
$$

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Posterior derivation

$$\Rightarrow \pi(\mu,\tau|Y) \propto \underbrace{\tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(n\tau+\kappa_0\tau)\left[\mu - \frac{(n\tau\bar{y}+\kappa_0\tau\mu_0)}{(n\tau+\kappa_0\tau)}\right]^2\right\}}_{\text{Normal Kernel}}$$

$$\times \underbrace{\tau^{\frac{\nu_0+n}{2}-1} \exp\left\{-\frac{\tau}{2}\left[\nu_0\sigma_0^2 + s^2(n-1) + \frac{n\kappa_0}{(\kappa_0+n)}(\bar{y}-\mu_0)^2\right]\right\}}_{\text{Gamma Kernel}}$$

$$= \underbrace{\tau^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\tau(\kappa_0+n)\left[\mu - \frac{(\kappa_0\mu_0+n\bar{y})}{(\kappa_0+n)}\right]^2\right\}}_{\text{Normal Kernel}}$$

$$\times \underbrace{\tau^{\frac{\nu_0+n}{2}-1} \exp\left\{-\frac{\tau}{2}\left[\nu_0\sigma_0^2 + s^2(n-1) + \frac{n\kappa_0}{(\kappa_0+n)}(\bar{y}-\mu_0)^2\right]\right\}}_{\text{Gamma Kernel}}$$

Posterior derivation

$$\Rightarrow \pi(\mu,\tau|Y) \;=\; \mathcal{N}\left(\mu_n, \frac{1}{\kappa_n\tau}\right) \times \text{Gamma}\left(\frac{\nu_n}{2}, \frac{\nu_n\sigma_n^2}{2}\right) = \pi(\mu|Y,\tau)\pi(\tau|Y),$$

where

$$\begin{aligned}
\kappa_n &= \kappa_0 + n \\
\mu_n &= \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_n} = \frac{\kappa_0}{\kappa_n}\mu_0 + \frac{n}{\kappa_n}\bar{y} \\
\nu_n &= \nu_0 + n \\
\sigma_n^2 &= \frac{1}{\nu_n}\left[\nu_0\sigma_0^2 + s^2(n-1) + \frac{n\kappa_0}{\kappa_n}(\bar{y}-\mu_0)^2\right] = \frac{1}{\nu_n}\left[\nu_0\sigma_0^2 + \sum_{i=1}^{n}(y_i-\bar{y})^2 + \frac{n\kappa_0}{\kappa_n}(\bar{y}-\mu_0)^2\right]
\end{aligned}$$

- Turns out that the marginal posterior of μ, that is, $\pi(\mu|Y) = \int_0^\infty \pi(\mu,\tau|Y)d\tau$ is a **t-distribution.**

- You can derive that distribution if you are interested, we won't spend time on it in class. We will be able to sample from it through Monte Carlo anyway.

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What's next?

Move on to the readings for the next module!

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